Math 132: Differential Topology

§ Hopf degree theorem

Let M, N be compact, connected, oriented manifolds of the same dimension.
The <u>degree</u> of a map $f: M \to N$, which counts the number of preimages (with sign) of a generic point in N, is a homotopy invariant, as we've seen.

In general, the degree is not a complete invariant, but it turns out to be the case when $N = S^n$:

<u>Thm</u> (Hopf degree theorem)
Let M be a compact, connected, oriented m-manifold.
Then, two maps $f_0, f_1 : M \longrightarrow S^m$ are homotopic if and only if

$$\deg f_0 = \deg f_1 .$$

proof) We already know ($\Rightarrow$).
For the other direction ($\Leftarrow$), set $W = M \times I$, and let $f: \partial W \to S^m$ be f_0 on $M \times \{0\}$ and f_1 on $M \times \{1\}$, so that $\deg f = \deg f_1 - \deg f_0 = 0$.

The claim then follows from the following more general theorem:

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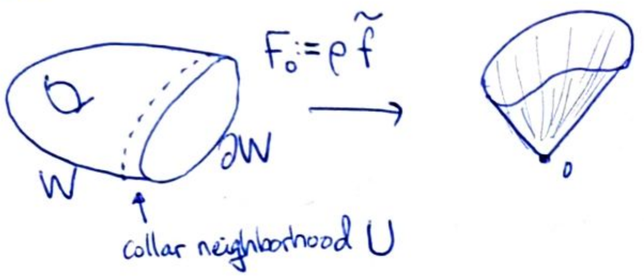
Thm (extension theorem)

Let W be a compact, connected, oriented $(m+1)$-manifold with boundary, and let $f : \partial W \to S^m$ be a smooth map.

Then, f extends to a map $F : W \to S^m$ with $\partial F = f$, if and only if $\deg f = 0$.

proof) We already know ($\Rightarrow$), so let's show ($\Leftarrow$).

Firstly, any smooth map $f : \partial W \to \mathbb{R}^{m+1}$ (without any degree constraint) can be extended to all of W:



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$\tilde{f} : U \xrightarrow{\pi} \partial W \xrightarrow{f} \mathbb{R}^{m+1}$,

$\rho : W \to \mathbb{R}$

$\rho|_{\partial W} \equiv 1$, $\rho \equiv 0$ outside a compact subset of U.

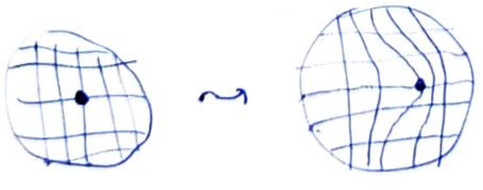
After some homotopy in the interior, we may assume that $0 \in \mathbb{R}^{m+1}$ is a regular value of F_0.

Using

Lemma (isotopy lemma)

Given any two points x, y in a connected manifold M, there exists a diffeomorphism $h : M \to M$ such that $h(x) = y$ which is isotopic to the identity. The isotopy may be taken to be compactly supported.

(each h_t is a diffeo)



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we may place the the finite set of points $F_0^{-1}(0)$ inside a ball $B \subset \mathrm{Int}(W)$.

By our assumption that $\deg f = 0$, $F_0|_{\partial B} : \partial B \to \mathbb{R}^{m+1} \setminus \{0\}$ has winding number 0 with respect to $0 \in \mathbb{R}^{m+1}$, i.e. $\deg\left(\frac{F_0}{|F_0|}\Big|_{\partial B} : \partial B \to S^m \right) = 0$.

In order to show that $\frac{F_0}{|F_0|}\Big|_{W - \mathrm{int}(B)} : W - \mathrm{int}(B) \to S^m$ extends to all of W, it suffices to prove the following special case of Hopf degree thm:

<u>Thm</u> Any smooth map $f : S^m \to S^m$ of degree 0 is homotopic to a constant map.

proof) We proceed by induction on m.

The case $m=1$ is an exercise (lift the map to a map $\mathbb{R} \to \mathbb{R}$).

Now, assume the claim is true for $m-1$, and we'll extend it to m.

Let $p \in S^m$ be a regular value. Using the isotopy lemma, we may assume that the preimage points $f^{-1}(p)$ are contained inside a ball $B \subset S^m$ such that $q \notin f(B)$ for some $q \in S^m$.

Then, we have

some diffeo sending p to $0 \in \mathbb{R}^m$

$$S^{m-1} \cong \partial B \xrightarrow{f|_{\partial B}} S^m \setminus \{q\} \cong \mathbb{R}^m ,$$

and it has winding number 0 around $0 \in \mathbb{R}^m$, since $\deg f = 0$.

From the induction hypothesis, $f|_{\partial B}$ is homotopic to a constant map, without crossing p.

Therefore, $f|_{S^m \setminus B} : S^m \setminus B \to S^m \setminus \{p\}$ extends to whole S^m, and thus $f : S^m \to S^m$ is homotopic to a constant map. ∎

Here's a nice application of the Hopf degree theorem:

<u>Thm</u> A compact, connected, oriented manifold M possesses a nowhere-vanishing vector field if and only if $\chi(M) = 0$.

proof) From Poincaré-Hopf, we already know $(\Rightarrow)$.

For the converse $(\Leftarrow)$, take a generic vector field v on M with finitely many zeros.

Using the isotopy lemma, we may place all the zeros in some ball $B \subset M$, and since $\chi(M) = 0$, the sum of indices of the zeros $= 0$.

It follows that the map $S^{m-1} \cong \partial B \xrightarrow{v} \mathbb{R}^m \setminus \{0\}$ has winding number 0 around $0 \in \mathbb{R}^m$, and by Hopf degree theorem, it is homotopic to a constant map.

Using this homotopy, we can extend $v|_{M \setminus B}$ to a nowhere-vanishing vector field on all of M. $\blacksquare$